

DUCKS IN SPACE

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Abstract. A subcritical pattern-forming system with nonlinear advection in a bounded domain is recast as a slow-fast system in space and studied using a combination of geometric singular perturbation theory and numerical continuation. Two types of solutions describing the possible location of stationary fronts are identified, one of which is present for all values of the bifurcation parameter while the other is present for zero or sufficiently small inlet boundary conditions but only when the bifurcation parameter is large enough. For slightly larger inlet boundary condition a continuous transition from one type to the other takes place as the bifurcation parameter increases. The origin of the two solution types is traced to the onset of convective and absolute instability on the real line. The role of canard trajectories in the transitions between these states is clarified and the stability properties of the resulting spatial structures are determined. Front location in the convective regime is highly sensitive to the upstream boundary condition and its dependence on this boundary condition is studied using a combination of numerical continuation and Monte Carlo simulations of the partial differential equation. Statistical properties of the system subjected to random or stochastic boundary conditions are interpreted using the deterministic slow-fast spatial-dynamical system.

Key words. Spatial dynamics, slow-fast systems, canards, convective and absolute instability

AMS subject classifications.

1. Introduction. There has been a great deal of interest in slow-fast systems in recent years [12]. Among the many phenomena such systems exhibit is the canard phenomenon [1, 22]. This term is used to refer to the abrupt growth of a periodic orbit or limit cycle that occurs in an exponentially narrow parameter interval. Within this interval the limit cycle follows for a time a repelling slow manifold. Phenomena of this type are invariably studied in the time domain and in this paper we seek to extend the slow-fast decomposition to solutions that evolve in space forming, for example, a front separating two distinct states of the system. We also describe stability properties of such states in *time*, in other words, we are interested in canard-like phenomena in a partial differential equation (PDE) context. Amongst the few studies in this direction, we mention *canard traveling waves*, which have been analysed in a moving frame as homoclinic connections with a canard segment [3, 17, 28], shock-like structures in PDEs which can also be interpreted in terms of canards [30, 4] and canard-related bifurcation delay in reaction-diffusion PDEs [14, 21]. We leave the study of temporal canards in PDEs to future work.

For this purpose we select a van der Pol-like example first studied by Chomaz & Couairon [7, 8]:

$$(1.1) \quad A_t = A_{xx} - (U - \alpha A^2)A_x + \mu A - A^3.$$

Here A represents a real variable with the bifurcation parameter μ controlling the stability of the trivial state $A_* \equiv 0$: homogeneous solutions grow when $\mu > 0$ and saturate at amplitude $A = \pm\sqrt{\mu}$ owing to the nonlinear term. When $\alpha = 0$ the

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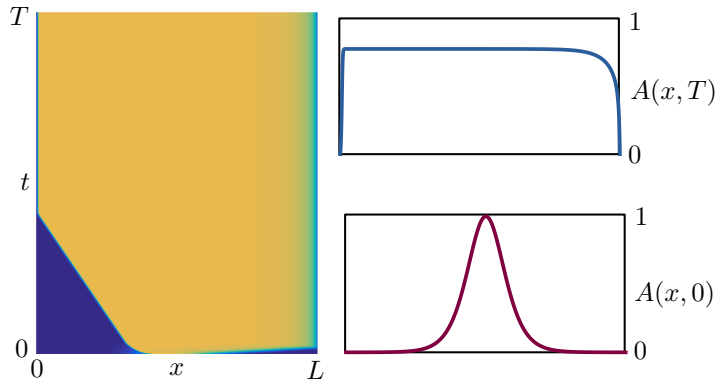


FIG. 1. Time simulation of (1.1) with Dirichlet boundary conditions $A(0, t) = A(L, t) \equiv 0$. Left: space-time plot for $(x, t) \in [0, L] \times [0, T]$, where $L = 10\pi$ and $T = 30$. Right: initial and final profiles of the time simulation. Parameters: $\mu = 7.4 < \mu_a$, $\alpha = 5$, $U = 12$.

advection term UA_x represents advection of spatial inhomogeneities to the right at speed U . As a result near $\mu = 0$ infinitesimal spatially localized disturbances travel to the right more rapidly than they grow and at each fixed location the disturbance decays to zero as $t \rightarrow \infty$. Thus, despite the instability of $A = 0$ in an appropriately moving frame, the state $A = 0$ is stable in this sense. In the fluid mechanical literature the point $\mu = 0$ is referred to as the threshold for convective instability. The term $\alpha > 0$ does not change this linear picture but it does imply that the advection speed is reduced as the amplitude of the state grows. Thus nonlinear states are not necessarily advected downstream and can in fact invade the domain (Figure 1). In the following we refer to this behavior as nonlinear absolute instability by analogy with the notion of (linear) absolute instability which arises at $\mu = \mu_a \equiv U^2/4$, the parameter value at which the growth rate of the instability is able to compete for the first time with the rate at which a perturbation is swept downstream, i.e., at $\mu = \mu_a$ the upstream spreading speed exactly balances the downstream advection. Note that on the real line steady states moving with speed c to the right obey the van der Pol ordinary differential equation (ODE)

$$(1.2) \quad A_{\xi\xi} + [\alpha A^2 - (U - c)]A_{\xi} + \mu A - A^3 = 0$$

with time replaced by the comoving spatial coordinate $\xi \equiv x - ct$. Periodic solutions of this equation with period ℓ travel at speed $c(\ell, \mu)$ determined as a nonlinear eigenvalue of the above problem.

It is clear that in systems of this kind boundaries play a crucial role. When boundary conditions are imposed x can no longer be considered as a time-like variable, although as shown below a spatial dynamics analysis continues to provide useful information. Linear theory shows that in the presence of (nonperiodic) boundary conditions that preserve the base state $A = 0$ this state loses stability at μ_G corresponding to the presence of a neutrally stable global eigenmode $A_G(x)$, where $\mu_G - \mu_a = O(L^{-2})$ and L is the length of the domain [8, 29]. Thus instability is delayed, by an $O(1)$ amount, from $\mu = 0$ to the vicinity of the onset of absolute instability (Figure 2(a)). The eigenmode $A_G(x)$ is typically compressed against the downstream wall (we call it a wall mode); as μ increases an abrupt transition takes place that fills the domain with the $A \neq 0$ state, except for a residual defect near the upstream boundary [29].

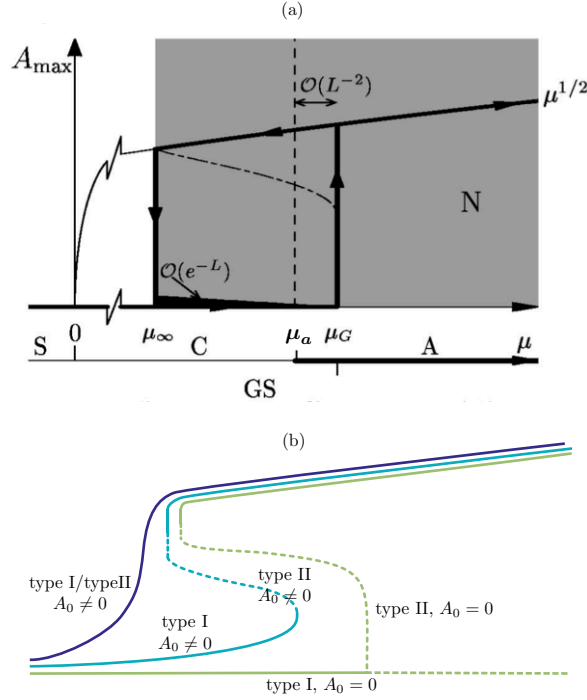


FIG. 2. (a) Bifurcation diagram for Eq. (1.1) when $U = 12$ and $\alpha = 1$ and zero inlet boundary condition. The location μ_a represents the onset of absolute instability on the real line. From [8]. (b) Sketch of a similar bifurcation diagram with solution branches for both zero or nonzero inlet boundary conditions showing the location of type I and type II solutions defined in the text.

The above scenario appears to be typical of systems with broken reflection symmetry, confined in a finite domain, provided the primary bifurcation is supercritical. However, as already mentioned, problem (1.1) is expected to become strongly subcritical when α is sufficiently large (in fact, $\alpha > 6/U$ [8]) and this fact suggests that a different domain-filling scenario may take place. Figure 2 shows a typical bifurcation diagram obtained for $\alpha > 6/U$ illustrating these ideas. We emphasize in particular the abrupt drop in amplitude of the solution near $\mu = \mu_\infty$ from an $O(\sqrt{\mu})$ amplitude to an apparently exponentially small amplitude. In the present paper we confirm these results and provide an explanation for the observed behavior in terms of the geometric theory of dynamical systems. In particular, we describe in detail the solution branch shown in Figure 2(a), hereafter referred to as a type II branch or equivalently as the absolute instability branch. In addition, we uncover a new class of solutions that are present for $\mu < \mu_G$ whenever the upstream boundary condition $A(0, t) \equiv A_0 \neq 0$. These are located on a type I branch or equivalently the convective instability branch which connects to a type II branch in different ways depending on the inlet boundary condition A_0 . Indeed, the transition from type I to type II solution branches can occur in one of three scenarios as sketched in Figure 2(b). First, when $A_0 = 0$, a hysteretic transition from the type I solution branch $A \equiv 0$ to the type II solution branch $A \neq 0$ takes place at a subcritical bifurcation at $\mu = \mu_G$. When $A_0 \neq 0$ but is “small enough”, the small amplitude type I states turn into type II states when the former undergo a saddle-node bifurcation at $\mu = \mu_{SN, \eta} < 0$. This transition also triggers a hysteretic jump to large amplitude type II states. For larger A_0 the

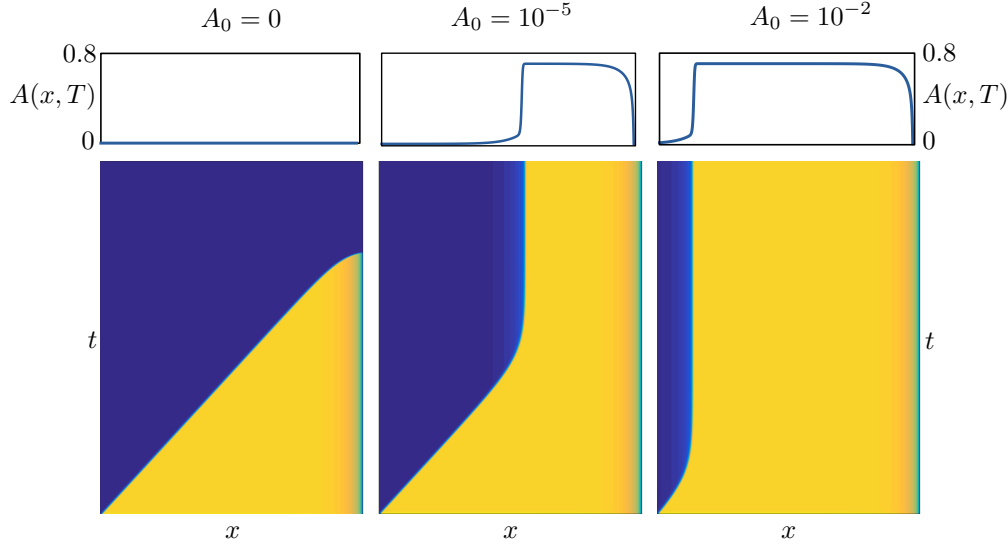


FIG. 3. Small variations of the inlet boundary condition $A(0, t) \equiv A_0$ have a dramatic impact on the steady states attained by system (2.1). Left: when we time-step (2.1) starting from a type IIb solution with zero inlet boundary condition, $A_0 = 0$, the system evolves towards the trivial steady state. Center: with $A_0 = 10^{-5}$ a type IIb solution is selected. Right: with $A_0 = 10^{-2}$, the final state remains a type IIb solution, but with a longer plateau. Parameters: $\varepsilon = 1/144$, $\mu = 7.4$, $\alpha = 5$, $(x, t) \in [0, 10\pi/\sqrt{\varepsilon}] \times [0, T]$, where $T = 30/\varepsilon$.

hysteresis disappears and the transition from type I states to type II states occurs without any stability change. Thus the bifurcation diagram in Figure 2(a) represents an important but singular case: the limiting bifurcation diagram in Figure 2(b) as $A_0 \rightarrow 0$.

The type I solutions are an analogue of the noise-sustained structures familiar from studies of convectively unstable systems [9] and hence play an important role in systems with random fluctuations at the inlet. In the present paper we quantify the precise role played by these solutions in determining the statistics of the location of the front that develops downstream from the inlet in the case of randomly distributed upstream boundary conditions. Figures 3 and 4 illustrate the sensitivity of the system in the convectively unstable regime to small changes in the inlet boundary condition.

2. PDE Stationary states. We study two regimes of the PDE (1.1), which we will denote as *system A* and *system B*, respectively.

2.1. System A. We consider Eq. (1.1) in the limit of large constant advection, large amplitude, short time scales and small spatial scales. For this purpose we adopt the scaling

$$0 < \varepsilon = U^{-2} \ll 1, \quad x \rightarrow \varepsilon^{1/2}x, \quad t \rightarrow \varepsilon t, \quad A \rightarrow \varepsilon^{-1/4}A,$$

and arrive at the PDE

$$(2.1) \quad A_t = A_{xx} - (1 - \alpha A^2)A_x + \varepsilon \mu A - \sqrt{\varepsilon} A^3, \quad (x, t) \in (0, L) \times (0, T).$$

This equation is to be solved subject to the Dirichlet boundary conditions

$$(2.2) \quad A(0, t) = A_0, \quad A(L, t) = 0, \quad t \in [0, T],$$

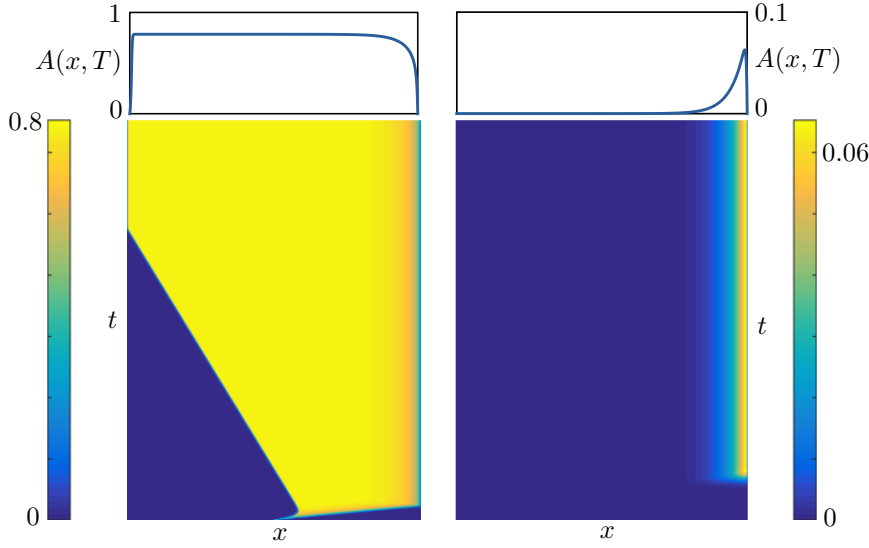


FIG. 4. Bistability in system A, Eq. (2.1), with $A_0 = 10^{-10}$, starting from two different initial conditions with the same control parameters. Other parameters as in Figure 3.

starting from a suitably-defined initial condition. The linear stability of steady states A_* of (2.1) is inferred by studying the eigenvalue problem $\lambda\psi = \mathcal{L}(A_*)\psi$, where

$$\mathcal{L}(A_*) = \partial_x^2 - (1 - \alpha A_*^2)\partial_x + 2\alpha A_*\partial_x A_* + \varepsilon\mu - 3\sqrt{\varepsilon}A_*^2$$

with null boundary conditions.

2.2. System B. System A captures large amplitude solutions to (1.1). However, it is also possible to consider solutions with $O(1)$ amplitude by studying the large nonlinear advection, short time, small spatial scales limit of (1.1),

$$0 < \varepsilon = U^{-2} \ll 1, \quad \alpha \rightarrow \varepsilon^{-1/2}\alpha, \quad x \rightarrow \varepsilon^{1/2}x, \quad t \rightarrow \varepsilon t,$$

leading to the PDE

$$(2.3) \quad A_t = A_{xx} - (1 - \alpha A^2)A_x + \varepsilon\mu A - \varepsilon A^3, \quad (x, t) \in (0, L) \times (0, T),$$

subject to the boundary conditions (2.2). Linear stability analysis for steady states of (2.3) is carried out as for system A.

3. Bifurcation diagram for system A. We track steady states of (2.1), subject to boundary conditions (2.2) using standard numerical continuation methods (see Appendix for the underlying PDE discretization). The corresponding bifurcation diagrams are presented in Figure 5 using the solution measure $S(A) = -\text{sgn } A'(L)\|A\|_2$, where the prime denotes differentiation with respect to x and $\|\cdot\|_2$ is the standard L_2 -norm. Only selected branches are shown.

For $A_0 = 0$, the boundary value problem associated with steady states of (2.1)-(2.2) is left invariant by the transformation $A \rightarrow -A$. The trivial state loses stability at $\mu = \mu_G \approx 36.030$, where a symmetry-breaking bifurcation gives rise to two branches of type II solutions. Such branches are initially unstable and restabilize at a saddle-node bifurcation. A second symmetry-breaking instability at $\mu \approx 37.708$ gives rise to branches of unstable solutions which change sign (see profiles 3 and 6 in Figure 5(a)).

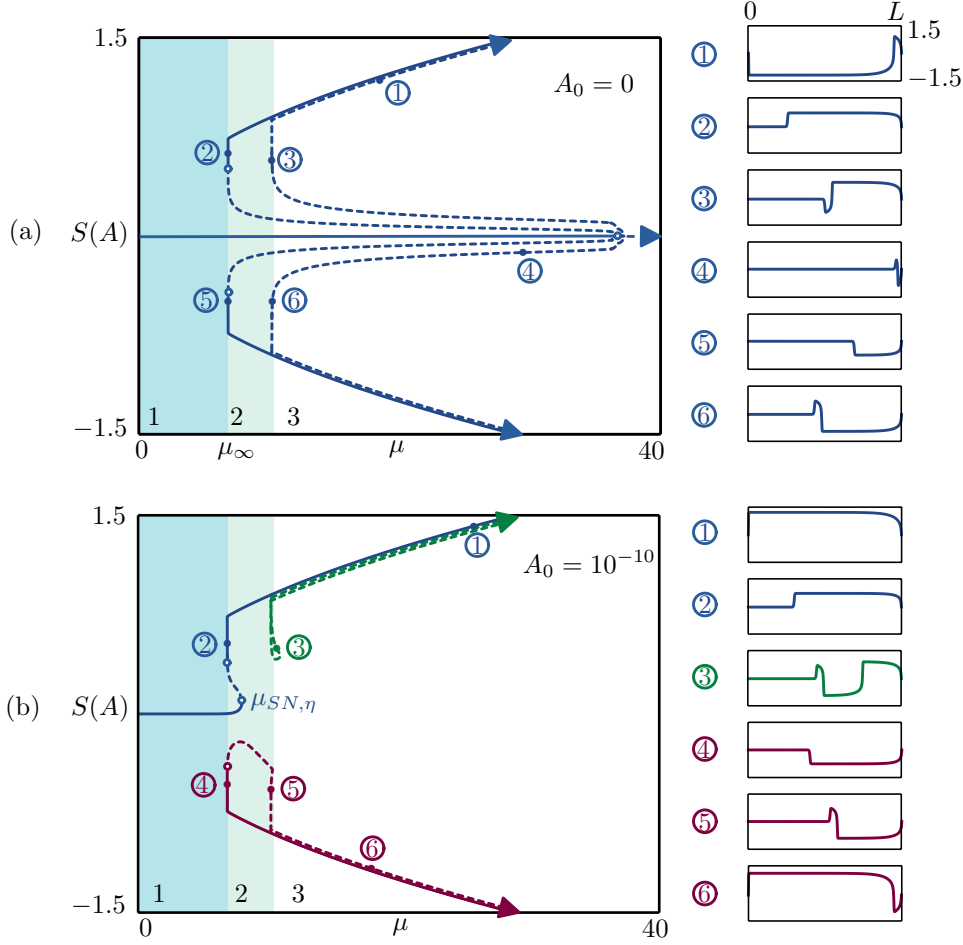


FIG. 5. Bifurcation diagram of steady states of system A with inlet boundary condition (a) $A_0 = 0$ and (b) $A_0 = 10^{-10}$ using the solution measure $S(A) = -\text{sgn } A'(L)\|A\|_2$. All solution profiles (right panels) are represented on $[0, L]$ with $A \in [-1.5, 1.5]$. Changes of stability are indicated with open circles and the locations of the solution profiles on the right are denoted using inscribed integers. Only selected solution branches are shown. Steady states in the μ intervals labeled 1–3 can be understood in terms of the spatial phase planes in Figure 6. Parameters in (a) are as in Figure 3, while parameters in (b) are as in Figure 4.

For vanishingly small inlet boundary conditions, $A_0 = 10^{-10}$, the branches perturb in a complex fashion. Branches of type II solutions with $A(0) = A_0$ still exist (profiles 2 and 4 in Figure 5(b)), as evidenced by the blue ($S > 0$) and red ($S < 0$) branches in Figure 5(b): the branch with $S > 0$ is initially stable and undergoes two saddle-node bifurcations, inducing bistability between steady states with different amplitudes and spatial localization (thereby explaining the numerical simulations reported in Figure 4); the other type II branch (red, with $S < 0$) is generated by a collision between branches containing profiles 5 and 6 in Figure 5(a) and also possesses a stable segment. The remaining branch in Figure 5(b) (green, $S > 0$) is fully unstable and derives from a collision with further branches of solutions whose sign changes multiple times (see profile 3 in Figure 5(b)).

4. The spatial van der Pol oscillator. It is advantageous to interpret the steady state problems associated with (1.1) as a second-order van der Pol equation with independent variable x . We refer to the resulting system as a *spatial van der Pol oscillator*. When considering system A (2.1), we set $z_1 = A_*$ and use the standard Liénard transform $z_2 = -A'_* + A_* - (\alpha/3)A_*^3$ to obtain the first-order system

$$(4.1) \quad z'_1 = -z_2 + z_1 - \frac{\alpha}{3}z_1^3$$

$$(4.2) \quad z'_2 = \sqrt{\varepsilon}(\sqrt{\varepsilon}\mu z_1 - z_1^3).$$

We identify steady states of system A, Eq. (2.1), with the solutions to the two-point boundary value problem (BVP) (4.1)–(4.2) with the boundary conditions

$$(4.3) \quad z_1(0) = z_{10} \equiv A_0, \quad z_1(L) = 0.$$

Similarly, the spatial dynamical formulation of (2.3) is given by

$$(4.4) \quad z'_1 = -z_2 + z_1 - \frac{\alpha}{3}z_1^3$$

$$(4.5) \quad z'_2 = \varepsilon(\mu z_1 - z_1^3).$$

We anticipate that for finite ε the bifurcation diagrams for system A and system B are very similar (see below), and indeed we shall always be presenting numerical results for finite ε and system A, while interpreting them in terms of the slow-fast system B.

In the present section we interpret solution branches of Figure 5 (and similar plots) using the slow-fast structure of the spatial-dynamical systems (4.1)–(4.2) and (4.4)–(4.5). To this end, we shall temporarily depart from the corresponding PDE formulation and present phase planes and numerical solutions to the spatial boundary value problems. In this section we use the term *spatial stability* or simply *stability* to classify the asymptotic behavior of solutions to (4.1)–(4.2) and (4.4)–(4.5). The *PDE stability* of the corresponding steady states for systems A and B is determined as described in Section 2.

Figure 6 shows the phase plane for system A and various values of the principal bifurcation parameter μ , with z_1 plotted horizontally and z_2 vertically. Each panel shows the nullcline N , usually referred to as the *critical manifold* of the slow-fast system,

$$(4.6) \quad z_2 = z_1 - \frac{\alpha}{3}z_1^3.$$

together with the line $z_1 = 0$ along which $z'_2 = 0$. There are three fixed points, one at the origin $O : (z_1, z_2) = (0, 0)$ and the others at $P_{\pm} : (z_1, z_2) = \pm(\varepsilon^{1/4}\sqrt{\mu}, \varepsilon^{1/4}\sqrt{\mu}[1 - (\alpha/3)\varepsilon^{1/2}\mu])$. The former is unstable with eigenvalues $\lambda = (1/2)[1 \pm \sqrt{1 - 4\varepsilon\mu}] \approx 1, \varepsilon\mu$, while the latter are always saddles. The eigenvalues of O are real if $\mu < 1/(4\varepsilon)$ but complex if $\mu > 1/(4\varepsilon)$. The equality $\mu = 1/(4\varepsilon)$ defines the onset of absolute instability, i.e., $\mu = \mu_a$. This is because the two-point BVP with the boundary conditions $A(0) = A(L) = 0$, or equivalently

$$(4.7) \quad z_1(0) = z_1(L) = 0,$$

only has solutions when the spatial eigenvalues are complex but not otherwise. Note that in the vicinity of this point the slow-fast decomposition necessarily fails. However,

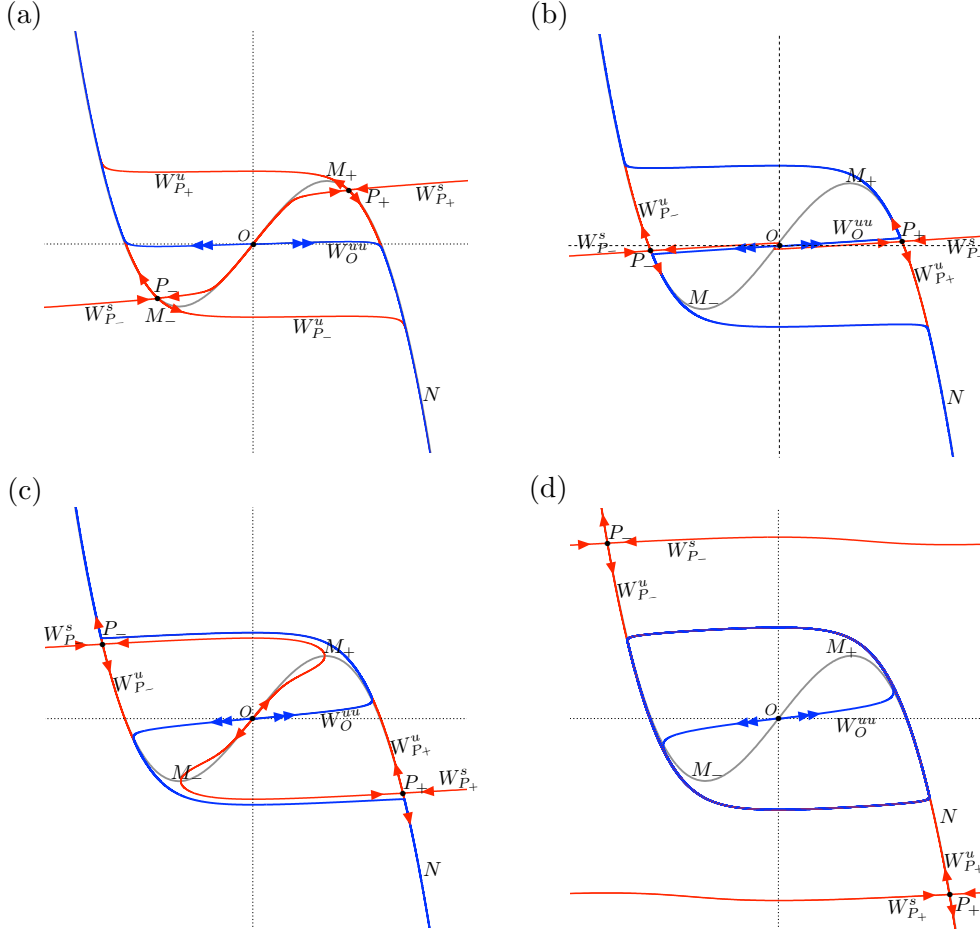


FIG. 6. Phase plane for system A, Eqs. (4.1)–(4.2), when $U = 12$, $\alpha = 5$, and (a) $\mu = 4$, (b) $\mu = 7$, (c) $\mu = 10$, (d) $\mu = 13$. The phase planes are representative of sectors 1–3 of Figure 5: (a) sector 1, (b) transition between 1 and 2, (c) sector 2, (d) transition between 2 and 3.

since this point moves off to infinity in the limit of interest, $\varepsilon \rightarrow 0$, this issue does not arise when studying solutions for $\mu = O(1/\sqrt{\varepsilon})$ or less.

Figure 6 shows that the critical manifold (fast nullcline) N has a pair of turning points $M_{\pm} : (z_1, z_2) = \pm(1/\sqrt{\alpha}, 2/3\sqrt{\alpha})$. In the following we are only interested in the saddle P_+ and its position on N relative to M_+ . We find that for $\mu < 1/(\alpha\sqrt{\varepsilon})$ the point P_+ lies between O and M_+ while for $\mu > 1/(\alpha\sqrt{\varepsilon})$ it lies past M_+ . In the former case P_+ is attracting along N (M_+ is repelling along N) while in the latter case it is repelling (M_+ is attracting along N ; see Figure 6). Moreover, the middle part OM_+ of N is repelling in the z_1 direction while that beyond M_+ is attracting. We also observe that P_+ lies on the axis $z_2 = 0$ when $\mu = (3/\alpha)\varepsilon^{-1/2}$.² These results suffice to construct the flow of system A in the (z_1, z_2) plane. Figure 6 shows several examples computed for the parameter values of Figure 5 (and similar to those used in [8]).

²It is an accident of the choice of parameters in [8] that this is the case precisely at $\mu = \mu_a$.

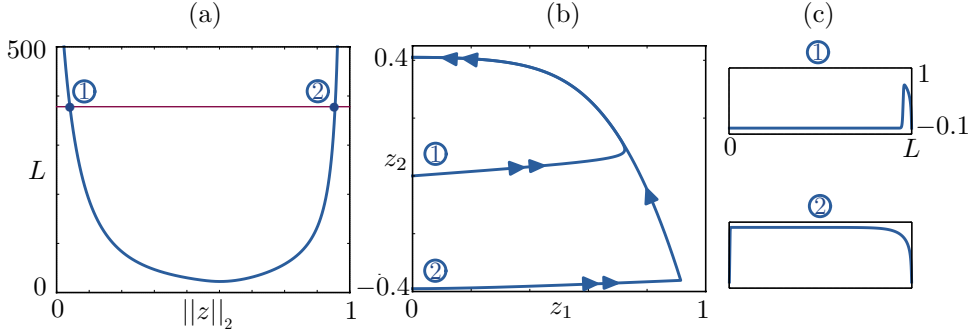


FIG. 7. (a) Solution branch obtained by varying L for fixed $\mu = 10$. Two solutions are found for $L = 377$ (red line), and these are shown as trajectories in the spatial phase plane (z_1, z_2) (panel (b)) and as spatial profiles $A(x)$ of the steady PDE (panel (c)). Parameters are as in Figure 6.

We now describe the solution of the BVP (4.7), i.e., the original boundary value problem of [7, 8], using the qualitative information contained in Figure 6. As we shall see, the phase portraits in Figure 6(a) are connected to sectors 1–3 of the bifurcation diagram in Figure 5.

We assume that $z_1(x) > 0$ for $x \in (0, L)$. Such solutions exist only if $z_2(0) < 0$. For suppose that $z_2(0) > 0$. Then z'_1 at $z_1 = 0$ is strictly negative, so that the flow enters immediately into the negative half plane. Hence solutions of this boundary value problem can exist only if $z_2(0) < 0$. We now argue that such solutions do not exist if μ is too small. The critical value of μ is determined by the so-called *orbit flip* bifurcation at which a branch of $W^s(P_+)$ coincides with the *strong unstable manifold* of O , denoted by $W^{uu}(O)$. As shown by Couairon & Chomaz [7] this occurs when $\mu = \mu_\infty = 3\alpha^{-1}(U - 3\alpha^{-1}) \approx 3/(\alpha\sqrt{\varepsilon})$. For $\mu > \mu_\infty$ the separatrix $W^s(P_+)$ must be below $W^{uu}(O)$. Hence, continued backwards in ‘time’, it must intersect the z_2 axis at some $z_{2,s} < 0$. It follows that for each $z_2 \in (0, z_{2,s})$ there exists $L > 0$ such that the BVP (4.7) has a solution with this choice of L and $z_2(0) = z_2$. Note that $L \rightarrow \infty$ both as $z_2(0)$ tends to 0 and to $z_{2,s}$ (Figure 7). It follows that for L large enough there exist at least two solutions of (4.7) for the same L , one with z_2 close to 0 and one with z_2 close to $z_{2,s}$. The solution with $z_2(0) \approx 0$ spends a long ‘time’ near O , which implies that its L_2 norm is small. The solution starting near $z_{2,s}$ spends a long time near P_+ . As μ approaches μ_∞ from above these two solutions approach each other and the solution with the small L_2 norm develops a longer segment near P_+ , implying growth of the L_2 norm.

As explained above, the BVP (4.7) has a pair of solutions for $\mu > \mu_\infty$, one of which disappears when its L_2 norm vanishes. This occurs, by definition, at $\mu = \mu_G$. Since $\mu_G \approx \mu_a$ for large L we conclude that these solutions are fundamentally a consequence of absolute instability of the trivial state $A \equiv 0$ that takes place at $\mu = \mu_a$. Both solutions start at $z_2(0) < 0$ and follow the fast dynamics to the vicinity of N above P_+ , then drift along N past M_+ , and finally return to the z_2 axis with $z_2(L) > 0$ (see Figure 7(b)). In the following we refer to solutions for which $z_2(0) \approx 0$ as type IIa, and to solutions for which the point $(0, z_2(0))$ is close to $W^s(P_+)$ as type IIb (we assume L is large). Note that no type II solutions are present for $\mu < \mu_\infty$. With the discussion above, we have explained the transition between sector 1 and 2 in Figure 5(a) using the slow-fast spatial system B.

We now introduce a small but generic perturbation of the BVP that results in a

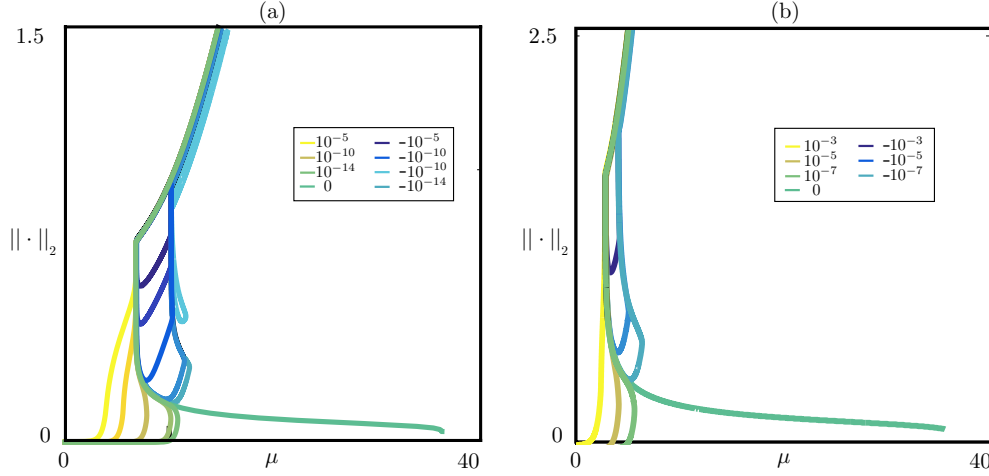


FIG. 8. Solution branches for different values of $z_1(0) \equiv \eta$ when $\alpha = 5$, $\varepsilon = 1/144$ in system A, Eqs. (4.1)–(4.2) (panel (a)), and when $\beta = 1$, $\varepsilon = 1/144$ in system B, Eqs. (4.4)–(4.5) (panel (b)). The panels show the type IIa,b branches for $z_1(0) = \eta \neq 0$, as well as the two type I branches; these differ in the sign of η (see legend) and are organized by an orbit-flip bifurcation at $\mu = \mu_\infty$ (see text). Note that for $\eta = -10^{-10}$ two solution branches are present; of these the right-most one corresponds to solution profiles with two turns (see Figure 13 and associated text for details).

new class of solutions:

$$(4.8) \quad z_1(0) = \eta, \quad z_1(L) = 0, \quad \text{with } 0 < \eta \ll 1.$$

We think of these boundary conditions as reflecting the presence of small amplitude imperfections or “noise” at the inlet. This perturbation has a number of consequences. It results in the presence of a nontrivial base state (called a type I solution), and introduces a reconnection between this state and the type IIa branch very close to $\mu = \mu_G$. In the following we refer to this fold as $\mu_{SN,\eta}$ (Figure 5); the location of this fold is extremely sensitive to the value of η , in contrast to the rest of the type I and II branches. Branch IIa then continues to larger amplitude and smaller μ towards the limit point bifurcation at μ_∞ (we ignore here the very weak dependence of μ_∞ on η) where it connects to the upper IIb branch. It follows that for the perturbed BVP (4.8), with η very small, there is an interval $[\mu_\infty, \mu_{SN,\eta}]$ of μ values, with $\mu_{SN,\eta} < \mu_G$, for which solutions of type II remain, and these solutions merge in a limit point bifurcation at μ_∞ (see Figures 7 and 8(a)). Since type I solutions still exist, it follows that for any fixed $\mu \in [\mu_\infty, \mu_{SN,\eta}]$ there are three different solutions to the BVP (4.8) as illustrated in Figure 9 (as before, we take L large).

However, we may have $z_2(0) > 0$ for solutions of type IIa, at least on some part of the interval $[\mu_\infty, \mu_{SN,\eta}]$. In addition, for each $\eta > 0$, no matter how small, there exists a solution of the BVP (4.8) with $z_2(0) > 0$ defined on the interval $\mu \in (0, \mu_{SN,\eta})$. These type I solutions (Figure 10) may contain canard segments, i.e., segments that follow the repelling part of N . Initially, when μ is small these solutions return very quickly to the z_1 axis and their canard segments are very short. However, as μ increases they contain a canard segment of increasing size. For μ close to $\mu_{SN,\eta}$ the solution of type IIa also develops a canard segment and finally at $\mu = \mu_{SN,\eta}$ the solutions of type I and type IIa merge in a limit point bifurcation at $\mu_{SN,\eta}$. For $\mu > \mu_{SN,\eta}$ there exists only one solution, corresponding to type IIb. Note that for

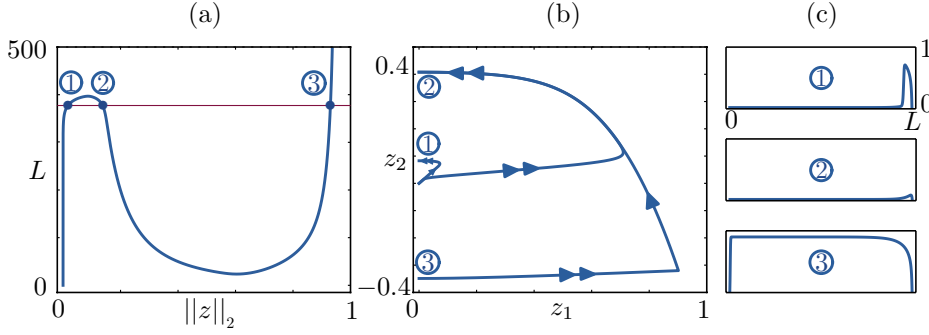


FIG. 9. (a) Solution branch obtained by varying L for fixed $\mu = 10$ and $\eta = 10^{-13}$. Three solutions are found for $L = 377$, shown in panels (b) and (c).

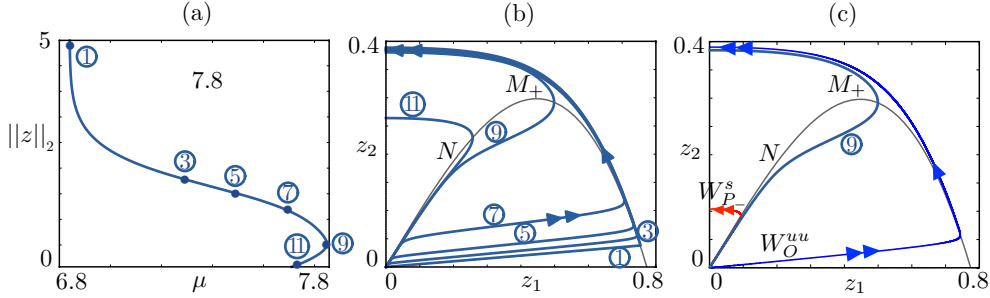


FIG. 10. (a) Detail of the type I solution branch with $\eta = 10^{-10}$ from Figure 8(a). (b) Portion of the phase plane showing the trajectories corresponding to the points indicated in (a). (c) Trajectory 9 from (a) and (b), displaying a (faux) canard segment, together with some of the invariant manifolds of equilibria of the system present at this value of μ . The point P_+ lies below the z_1 axis.

$\mu \in (\mu_\infty, \mu_{SN,\eta})$ there exist three distinct solutions (see Figure 8(a) for the smallest values of η). The presence of the limit point $\mu_{SN,\eta}$ is thus related to the transition from solutions analogous to canard cycles with no head to ones analogous to canard cycles with a head (Figure 11 and [13]). For larger η the limit point $\mu_{SN,\eta}$ disappears and solutions of type I change by homotopy to solutions of type II.

Figure 10(a) shows a detail of the branch corresponding to $\eta = 10^{-10}$ in Figure 8(a). The numbers 1–11 refer to points along this branch and the corresponding phase portraits are shown in Figure 10(b). Note that in contrast to the canard explosions that occur in exponentially small intervals in two-dimensional systems [1], in the present case no canard explosion takes place and canard behavior takes place over an $O(1)$ interval of values of the parameter μ . This is due to the fact that the dependence of the flight time along N on μ is much more regular than the dependence on the distance between the attracting and the repelling slow manifolds, which is the quantity governing the existence and size of canard cycles.

We next introduce a different perturbation of the BVP (4.7),

$$(4.9) \quad z_1(0) = -\eta, \quad z_1(L) = 0, \quad \text{with } 0 < \eta \ll 1.$$

Branches of solutions of (4.9) are shown in Figure 8(a) and in close-up in Figure 11. Figure 11 shows the corresponding solution profiles. Below we describe the solutions labeled 0, 1, \dots in Figure 11. Solution labeled 0 is very similar to a type IIb solution

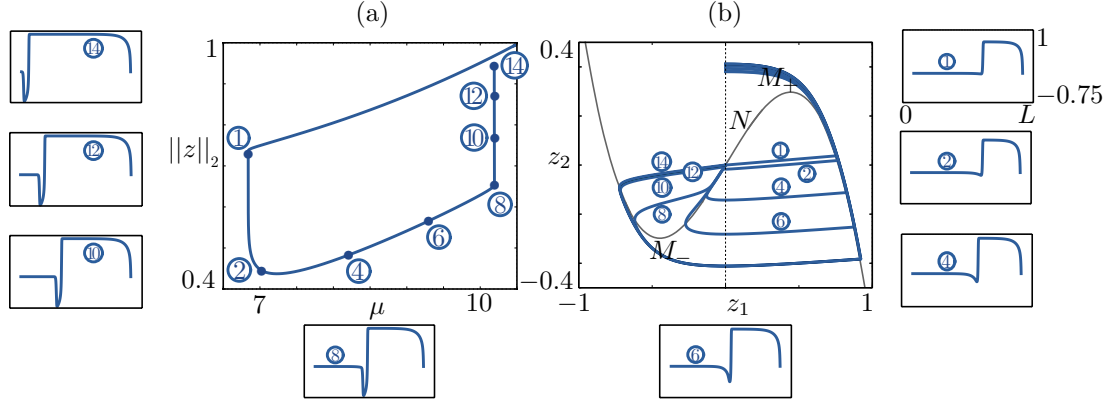


FIG. 11. (a) Detail of the type I solution branch with $\eta = -10^{-7}$ from Figure 8(a). (b) Portion of the phase plane showing the trajectories corresponding to the locations indicated in (a). Spatial profiles of the solutions of the BVP (4.9) corresponding to the trajectories shown in (b) are shown alongside. For the parameter values of Figure 8(a) $\mu_\infty \approx 6.84$ and $\mu_{c+} \approx 10.19$.

of (4.7): it starts close to an unstable fiber of N , jumps to the attracting part of N , follows N to M_+ and returns to the z_2 axis along a fast fiber. As μ decreases towards μ_∞ and P_+ moves up along N , all solutions are of this type. This is very similar to the situation in the BVP (4.8), since segments of these branches (for $\mu > \mu_\infty$) remain close to the corresponding solution branches IIa,b in BVP (4.7). The vertical segment between labels 1 and 2 corresponds to a passage near the orbit flip transition and along a segment of the branch of solutions of type IIb. The branch is similar to the branches of solutions of (4.8) as it follows the unperturbed solution branch corresponding to solutions of type IIb for μ up to a certain value that increases as η decreases. However, the relevant solutions of (4.9) differ from the corresponding solutions of (4.8) in that they leave the vicinity of the unperturbed branch ($z_1(0) = 0$) in the opposite direction (see Figure 8(a)). After that, the solutions start developing a canard segment and the branch has a fold that more-or-less corresponds to the maximal canard segment. The main difference from the branches corresponding to (4.8) is that canard segments with $z_1(0) = -\eta$ follow the repelling branch of N below the vertical axis; compare solutions 3 to 7 in Figure 11 with solutions 9 to 11 in Figure 10. Past the fold point, solutions of (4.9) with canard segments make an excursion towards the left attracting branch of N before jumping at M_- and following the right branch of N towards M_+ ; see solutions 8 to 14 in Figure 11. For the range of μ values where this occurs, the saddle point P_+ lies below M_- . As μ approaches the value of μ_{c+} , which corresponds to a connection from the maximal canard to P_+ (see Figures 11(b) and 12) the corresponding solutions, after jumping near M_- , come closer and closer to P_+ , which incurs a large increase of the L_2 norm for a very small change in μ . This argument explains the presence of the second vertical part of the branch seen in Figure 11(a) at $\mu = \mu_{c+}$. We mention that μ_{c+} is extremely close to the value $\mu = \mu_h$ at which there is, in the limit $\varepsilon \rightarrow 0$, a heteroclinic orbit connecting P_+ to P_- via the turning points M_\pm . However, the latter solutions are of no relevance to any of the BVP considered in the present work.

We may think of the type I solutions as small amplitude solutions and type II solutions as large amplitude solutions. The former originate in convective instability that sets in at $\mu = 0$ in the absence of boundaries while the latter arises from absolute

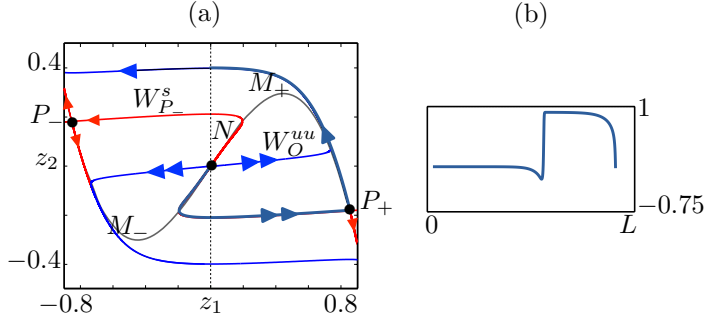


FIG. 12. (a) Trajectory 5 (in black), containing a (faux) canard segment, from Figure 11; also shown is a portion of the phase plane showing the repelling and attracting nullclines as well as stable and unstable manifolds of the three equilibria of the system present at this value of μ . (b) Spatial profile of the corresponding solution to the BVP (4.9).

instability. Both solutions exist when P_+ lies below the z_1 axis but as μ decreases and P_+ moves above the axis type II solutions are destroyed via a special solution that follows the strong unstable manifold of the origin to P_+ [7]. Solutions leaving O along this manifold are characterized by very rapid increase in amplitude, followed by a lengthy interval at amplitude $z_1 \approx \sqrt{\mu}$ before decaying back to $z_1 = 0$ at $x = L$. This condition determines the condition for an orbit flip; as already mentioned, in the limit $\varepsilon \rightarrow 0$, this condition is equivalent to the passage of P_+ through the z_1 axis, i.e., to $\mu = (3/\alpha)\varepsilon^{-1/2}$. When $\alpha = O(1)$ the location of this point moves to large values of μ ; however, in system B, this transition occurs at a finite value of μ , $\mu = 3/\beta$. In fact, in the definition of type I/type II solutions we can replace the z_1 axis by (one of the branches of) the strong unstable manifold $W^{uu}(O)$; for $\varepsilon \approx 0$ these two sets are very close to one another. Specifically, a solution is of type I if $(\eta, z_2(0))$ is above $W^{uu}(O)$ and of type II if $(\eta, z_2(0))$ is below $W^{uu}(O)$. A solution of type I or II which passes near the right branch of N exists only if $(\eta, z_2(0))$ is above $W^s(P_+)$. It follows that solutions of type I can be found as long as P_+ is below M_+ , while solutions of type II exist only if $\mu > \mu_\infty$, i.e., $W^s(P_+)$ is below $W^{uu}(O)$. This is consistent with the discussion in the preceding paragraphs.

In fact there is another class of solutions to the BVP, corresponding, in phase space, to one or more full turns around the origin before returning to the z_1 axis at $x = L$. These solutions follow the heteroclinic cycle mentioned above, and we can understand their behavior using the same phase plane analysis. We focus on the branch in the central panel of Figure 13. The properties of the solutions on this branch can be classified by their behavior near the origin and near the saddle points P_\pm . For example, the solution labeled 7 is at a fold point of the branch. Like solution 5 from Figure 12, it contains a maximal canard segment; the explanation of that fold point is similar to that provided earlier. Solutions 5 and 6 in Figure 13 occur after and before the transition through the maximal canard, respectively. Solutions 1 and 2 are located at local maxima on the branch and the associated μ values are both very close to the (right) vertical segment on the branch shown in Figure 11(a); as in the similar cases discussed earlier, these solutions come close to either of the two saddle points, explaining the sharp increase in their L_2 norm. Lastly, solution 3 is close to a type II solution, and solution 4 is very similar to such a solution but with $z_2(0)$ positive. Solutions with more turns are also possible and behave analogously.

Finally, since the transition from the branch with $\eta = 0$ to the branches with $\eta \neq 0$

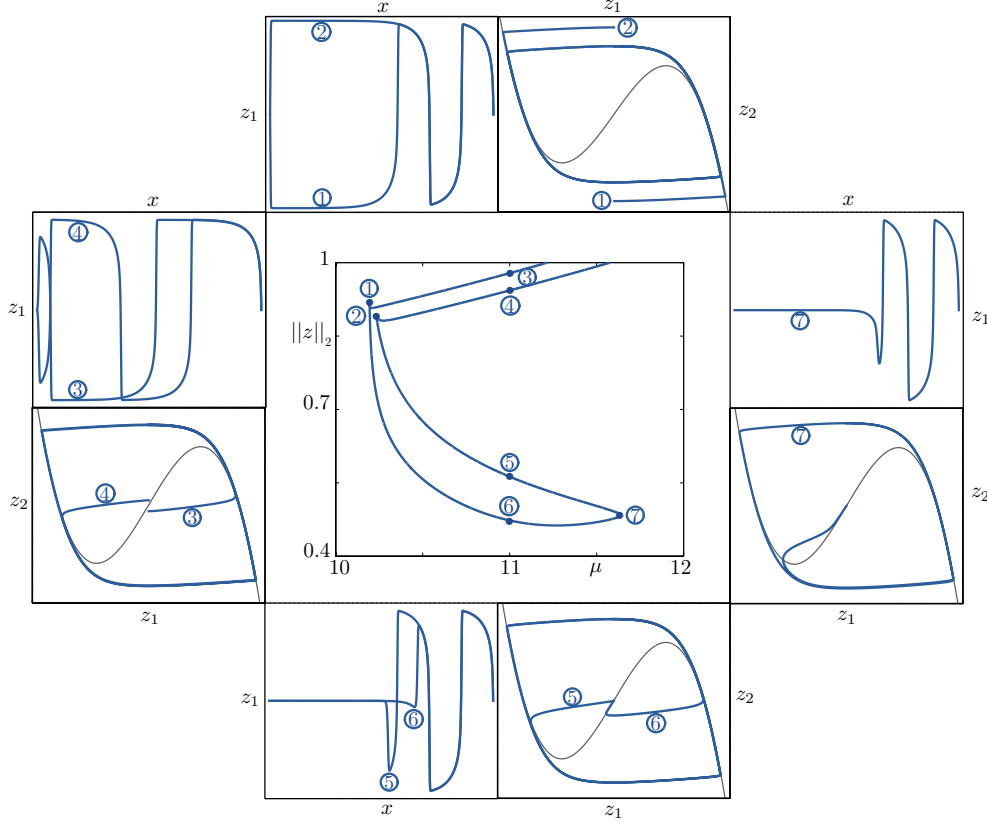


FIG. 13. Branch (central panel) corresponding to solutions that make a full turn around N before connecting back to $z_1 = 0$. The central panel shows a zoom of the rightmost branch from Figure 8(a) corresponding to $\eta = -10^{-10}$. Side panels show different solution profiles in space and in the phase plane at different locations along this branch as indicated by the inscribed integers.

can be seen as a broken pitchfork bifurcation, we expect to find disconnected solution branches for $\mu > \mu_G$ and $\eta \neq 0$. These are indeed present, as shown in Figure 14 for $\eta = 10^{-7}$ and $\eta = 10^{-5}$. The solution profiles on these branches correspond to orbit segments that make several turns along N before connecting back to $z_1 = 0$, and these lie, in the limit $\varepsilon \rightarrow 0$, on additional solution branches of the original BVP (4.7). In fact, for L sufficiently large, there will be many such solutions to the BVP considered by Chomaz & Couairon, depending on the number of turns they make along N before connecting back to $z_1 = 0$. When the inlet boundary condition becomes $z_1(0) = \eta \neq 0$ with η small (positive or negative), one finds solution branches that follow the unperturbed ($\eta = 0$) branches corresponding to a given number of turns; such branches may also connect two such unperturbed branches, as shown in Figure 14. Note that all the branches that exist in the range $\mu > \mu_G$ have regular limit points without vertical segments, in contrast to the branches passing through $\mu = \mu_\infty$ and $\mu = \mu_{c+}$. This is because for $\mu > \mu_G$ the unstable equilibrium point at the origin acquires complex eigenvalues (thereby preventing orbit flips) so that the long-term dynamics of the ODE system (4.1)–(4.2) is a stable limit cycle surrounding the origin (thereby preventing any connection between the origin and the saddle points P_\pm).

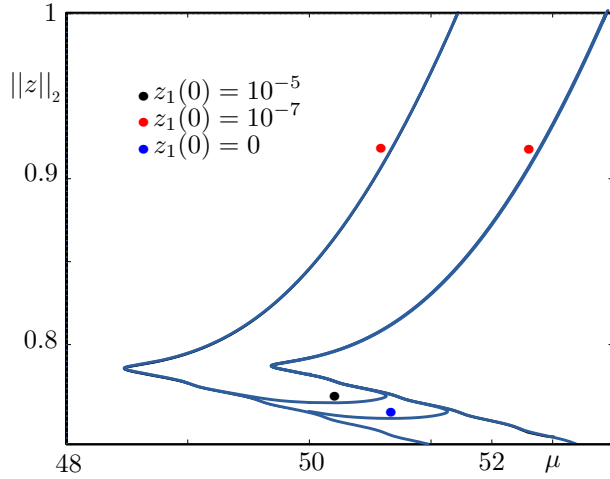


FIG. 14. Solution branches for $\mu > \mu_G \approx 36.030$, for both $\eta = 0$ and $\eta > 0$ (see legend). The solutions resemble those shown in Figure 13, but with more than one full turn along N . Unlike the type I and II branches described earlier these branches do not exhibit any vertical segments. This is a consequence of the fact that for $\mu > \mu_G$ the origin $(0, 0)$ has complex (spatial) eigenvalues.

5. PDE simulations. Given the sensitivity of the system to the inlet boundary condition $A_0 = \eta$, a question arises as to how solutions behave when this parameter is time-dependent or randomly distributed.

5.1. Solution profiles. Figure 15 shows the solution branches in the convective and absolute instability regimes together with sample solution profiles. The former are seen to be extremely sensitive to the inlet boundary condition $A(0, t) = \eta$ while the latter are not, as expected from the nature of the instability responsible for these branches.

Figure 15(b) compares the solution profiles along the two types of solution branches, starting with the profiles 1–3 along the absolute instability branch. The location of the solutions on the branch is indicated in Figure 15(a). Profile 3 shows a solution that is strongly compressed against the downstream boundary; this solution resembles the neutral wall mode predicted by linear stability theory for the state $A \equiv 0$. Decreasing μ leads to an increase in the amplitude of the solution (Figure 15(a)); at the same time the solutions broaden and starts to fill the domain. Profile 2 corresponds to $\mu = \mu_\infty$ and is located in the region of abrupt amplitude growth associated with the orbit flip. We see that the solution develops a plateau corresponding to $A = \sqrt{\mu_\infty}$ and that the length of this plateau increases as the solution norm increases. The shape of the front connecting this plateau to $A = 0$ remains fixed but the location of the front migrates upstream. This process terminates when the front reaches the upstream boundary and so becomes pinned at a particular location. Subsequent to this pinning event the branch turns around to larger values of μ and the amplitude of the plateau grows in proportion to $\sqrt{\mu}$ (profile 1).

Profiles 4–8 in Figure 15(b) are instead selected from the convective instability branches. Profiles 4–7 show the evolution of the solution with increasing μ when $\eta = 10^{-1}$. Near $\mu = 0$ the solution decreases from $\eta = 10^{-5}$ at $x = 0$ towards $A = 0$ at $x = L$. The slope is almost constant throughout the domain, indicating a balance between UA_x and μA with small adjustment near the downstream boundary

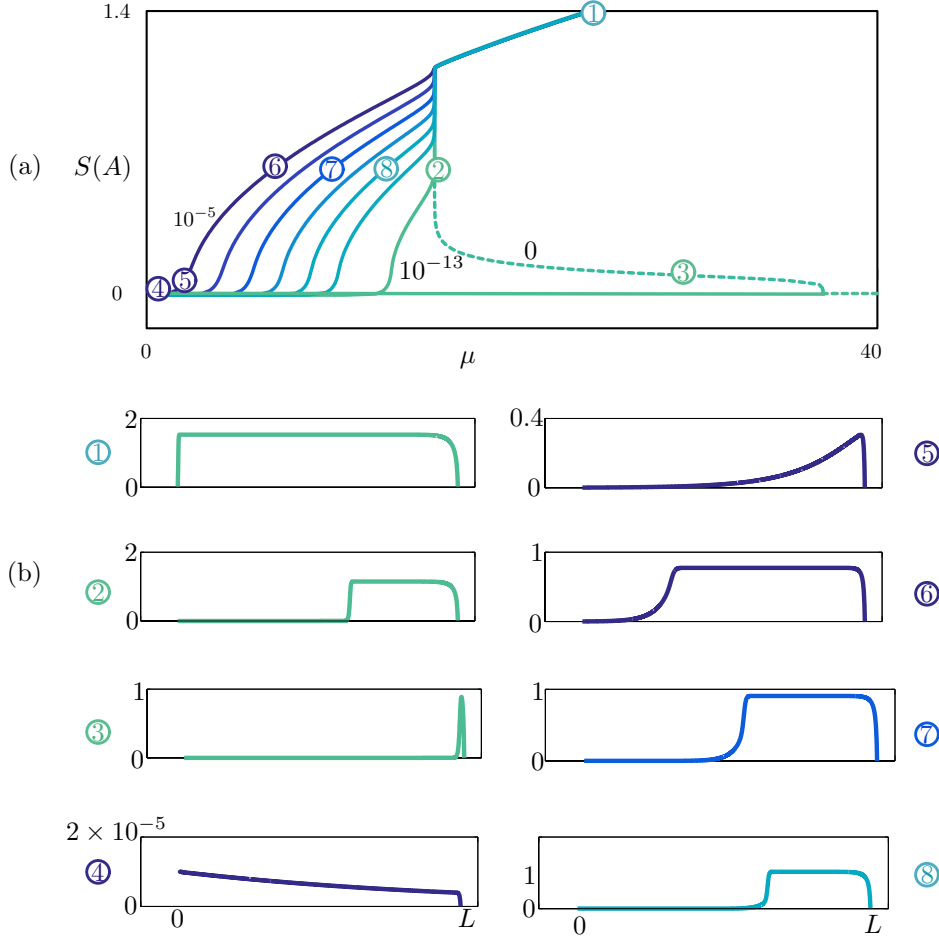


FIG. 15. (a) Branches of steady states for system A with $\alpha = 2$, $\varepsilon = 1/144$, $L = 10\pi/\sqrt{\varepsilon}$, and various inlet boundary conditions $A_0 \in [0, 10^{-5}]$, together with (b) the corresponding solution profiles $A(x)$, $x \in [0, L]$. The vertical scale is indicated for each profile.

(profile 4). As μ increases the slope reverses, with the amplitude growing towards the downstream boundary (profile 5). As μ increases further the amplitude near the downstream boundary saturates and the solution develops a plateau that evolves in a similar fashion to that on the absolute instability branch with increasing μ (profile 6). Profiles 7 and 8 correspond to branches with ever decreasing values of η . Since decreasing η delays the onset of spatial growth the solutions reach a given norm at larger μ and hence with a higher amplitude of the plateau.

5.2. Stability in time. A PDE stability calculation shows that the convective instability branches are stable in time throughout their range of existence when the fold at $\mu = \mu_{SN,\eta}$ is absent, and stable in $0 < \mu < \mu_{SN,\eta}$ when the fold $\mu_{SN,\eta}$ is present. In the former case stability is transferred at the degenerate fold to the upper part of the absolute instability branch; in the latter case there is a hysteretic transition to the upper absolute instability branch. Thus a stable solution is present for all $0 < \mu < \mu_G$, provided only that $A(0) \neq 0$.

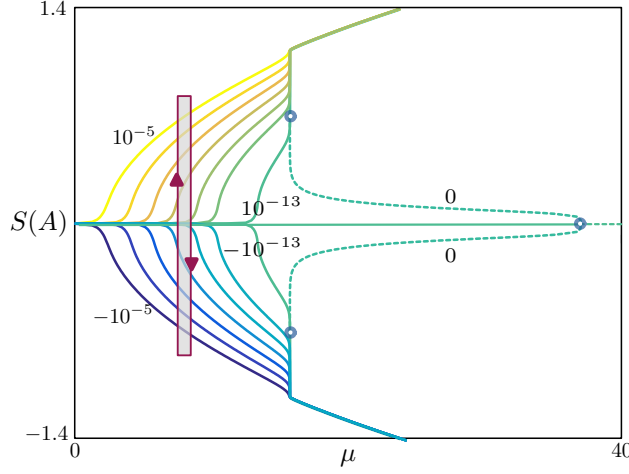


FIG. 16. Branches of steady states from Figure 15 for various inlet boundary conditions $A_0 \in [-10^{-5}, 10^{-5}]$ with a superposed cycle (in red) showing typical changes in the solution amplitude when A_0 oscillates quasistatically in time with mean $A_0 = 0$. A concrete example is shown in Figure 17.

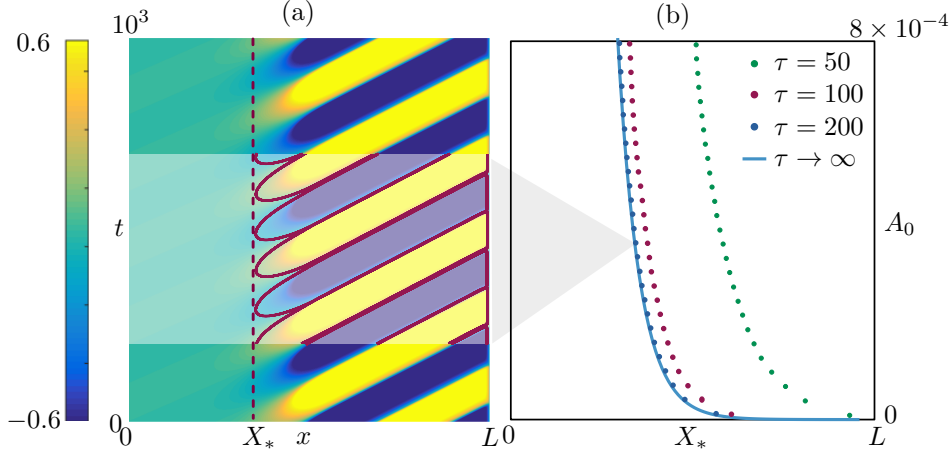


FIG. 17. (a) Location X_* from Eq. (5.1) of time-periodic fronts when system A is subjected to a time-periodic inlet boundary condition $A(0, t) = A_0 \sin(2\pi t/\tau)$; such fronts are sustained entirely by the small amplitude modulation in $A(0, t)$ (in the example shown here $\mu = 6.1$, $A_0 = 4 \times 10^{-4}$, $\tau = 200$, with the other parameters as in Figure 16). The red lines in the central part are the contour lines $A(x, t) = \pm 0.1$. (b) The front location X_* for various values A_0 and τ (dots). The steady states of Figure 16 have a well-defined X_* which we continue at $\mu = 6.07$ in the parameter A_0 , and use the resulting X_* as a solution measure (curve marked $\tau \rightarrow \infty$). The large τ time-periodic simulations are well approximated by the steady-state continuation.

5.3. Time-dependent inlet boundary condition. Figure 16 presents the solution branches computed for system A with $\alpha = 2$ and different values of the inlet amplitude A_0 . The results are similar to those presented in the previous sections for $\alpha = 5$: as A_0 varies within an $O(10^{-5})$ interval of $A_0 = 0$ the front location in the corresponding steady states changes dramatically, as witnessed by the $O(1)$ variations in the solution measure $S(A)$.

We now fix μ and prescribe time-periodic inlet boundary conditions, $A(0, t) = A_0 \sin(2\pi t/\tau)$. As sketched in Figure 16, for fixed μ and sufficiently large τ we expect the front location to oscillate as the spatio-temporal solution drifts from one branch of stable stationary states to the next and back again.

In Figure 17(a) we present a numerical experiment for $\mu = 6.07$ with a slowly-varying, small-amplitude inlet condition ($A_0 = 4 \times 10^{-4}$, $\tau = 200$). As expected, a time-periodic front establishes towards the centre of the domain; we remark that this front is sustained solely by the small periodic variations in the inlet boundary conditions: if $A_0 = 0$ the only attractive state in this region of parameter space is the trivial state $A_* \equiv 0$, as seen in Figure 16.

In order to quantify the position of the front, we monitor the level sets of $A(x, t)$ as t varies. In Figure 16 we define this position to be³

$$(5.1) \quad X_* = \min_{t \in [0, T]} \{x \in [0, L] : |A(x, t)| = 0.1\}.$$

We expect the steady state analysis presented in the past section to provide information about the front location, and this is confirmed by the results in Figure 17(b). The figure shows the results of steady state continuation in the parameter A_0 , using X_* as a solution measure (curve labeled $\tau \rightarrow \infty$ in Figure 17(a)): the solutions on this branch are steady states, but they have a well-defined X_* which approximates the location of time-periodic fronts like that in Figure 17(a). We repeated the direct numerical simulations for various values of A_0 and τ , and found that the steady-state approximation is excellent for slow oscillations of the inlet boundary conditions (Figure 17(b), $\tau = 200$), although it degrades as the oscillations become faster (Figure 17(b), $\tau = 50, 100$).

5.4. Uncertainty quantification of the front location. Motivated by the discussion in the previous sections, we consider steady states of the problem

$$(5.2) \quad \begin{aligned} A_t &= A_{xx} - (1 - \alpha A^2)A_x + \varepsilon \mu A - \sqrt{\varepsilon} A^3, & (x, t) &\in (0, L) \times (0, T), \\ A(0, t, \omega) &= A_0(\omega), \quad A(L, t, \omega) = 0, & t &\in [0, T], \\ A(x, 0, \omega) &= (1 - x/L)A_0(\omega), & x &\in [0, L], \end{aligned}$$

where the random inlet parameter $A_0(\omega) : \Omega \rightarrow \mathbb{R}$ has an associated density p_{A_0} . In what follows, we will choose p_{A_0} so as to give samples $|A_0| \ll 1$, that is, we prescribe small random inlet boundary conditions and study the propagation of uncertainty in the front location X_* of steady states attained from approximately flat initial conditions (note that the slope of the initial condition in (5.2) is negligible, since $|A_0| \ll 1$ and $L = O(\varepsilon^{-1/2})$).

Once again, the bifurcation diagrams of Figure 16 help us to understand the problem. We start by fixing $\mu < \mu_\infty = 15.75$. For a fixed event $\omega \in \Omega$, there is a single attracting steady state $A_*(x, \omega)$ with a front at $x = X_*$ that satisfies $|A_*(X_*(\omega), \omega)| = 0.1$. Moreover, for fixed $\omega \in \Omega$, the random variable X_* is a *deterministic* function of A_0 : the graph of such a function is approximated by the curve labeled $\tau \rightarrow \infty$ in Figure 17(b). This means that we can infer directly the density p_{X_*} using p_{A_0} and the bifurcation diagram. In Figure 18(a) we use numerical continuation to compute $X_*(A_0)$ in the interval $A_0 \in [-0.03, 0.03]$ for $\mu = 7, 12 < \mu_\infty$; we then prescribe

³Strictly speaking, the minimum is taken only over the subset of $[0, T]$ for which the threshold $A(x, \cdot) = 0.1$ is attained, for there may be times for which $A(x, t) < 0.1$ for all $x \in [0, L]$.

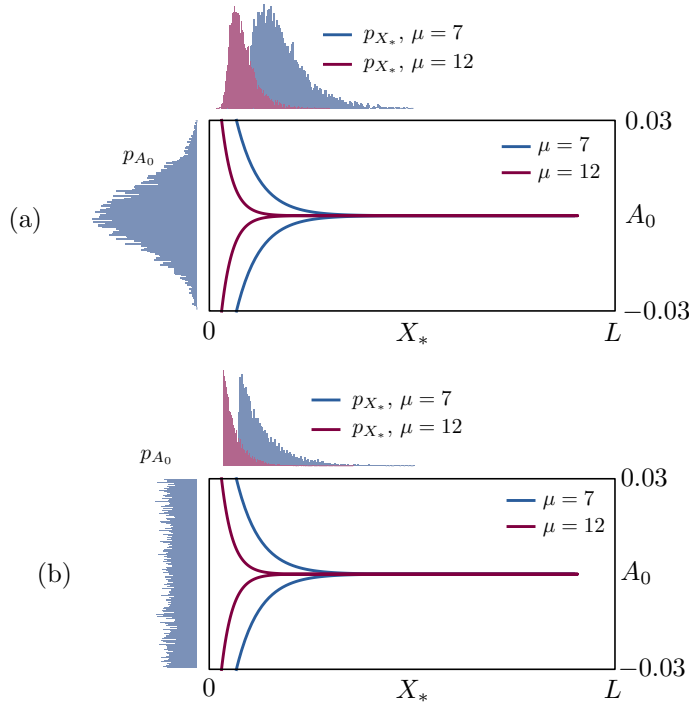


FIG. 18. For randomly distributed inlet boundary conditions A_0 with density p_{A_0} , the location X_* of the front in problem (5.2) is distributed with induced density p_{X_*} . Since for each sample there is a deterministic relation between A_0 and X_* (the blue and red branches $X_*(A_0)$), we infer the density p_{X_*} directly from $X_*(A_0)$ (upper panels). (a) If $p_{A_0} = \mathcal{N}(0, 0.01)$, then X_* is approximately Beta-distributed; we observe $\mathbb{E}[X_*] \rightarrow 0$, $\text{var}[X_*] \rightarrow 0$ as $\mu \rightarrow \mu_\infty$, i.e., the distribution of front locations moves towards the inlet as μ increases towards μ_∞ . (b) The experiment is repeated, with qualitatively similar results, for $p_{A_0} = \mathcal{U}(-0.03, 0.03)$. The distribution p_{X_*} is now approximately exponential. Parameters are as in Figure 16.

a normally distributed inlet boundary condition $p_{A_0} = \mathcal{N}(0, 0.01)$ and compute the corresponding densities p_{X_*} .

The histograms for p_{X_*} , obtained using 10^4 samples, show that the distribution of the front locations is shifted towards the inlet; we observe, in addition, that p_{X_*} resembles a Beta distribution with $\mathbb{E}[X_*] \rightarrow 0$, $\text{var}[X_*] \rightarrow 0$ as $\mu \rightarrow \mu_\infty$. Consequently, the uncertainty in the front location decreases considerably as we transition towards μ_∞ : for $\mu = 7$, some samples have fronts close to $L/2$, as can be seen in the blue histogram of Figure 18(a), while the distribution for $\mu = 12$ peaks much closer to the inlet. In Figure 18(b) we repeat the calculations for uniformly distributed inlet boundary conditions, $p_{A_0} = \mathcal{U}(-0.03, 0.03)$. The results are qualitatively similar to the normally distributed case, except that p_{X_*} is now approximated by an exponential distribution.

A complication arises when we consider the case $\mu > \mu_\infty = 15.7$. In this region of parameter space more than one attracting steady state may be present for each fixed event ω . We have not shown such branches in Figure 16, but refer the reader to Figure 5(b), where we show an example of two coexisting branches for $A_0 = 10^{-10}$ and $\alpha = 5$.

Figure 19 shows the results for $\mu = 17 > \mu_\infty$, i.e., in a region of parameter space

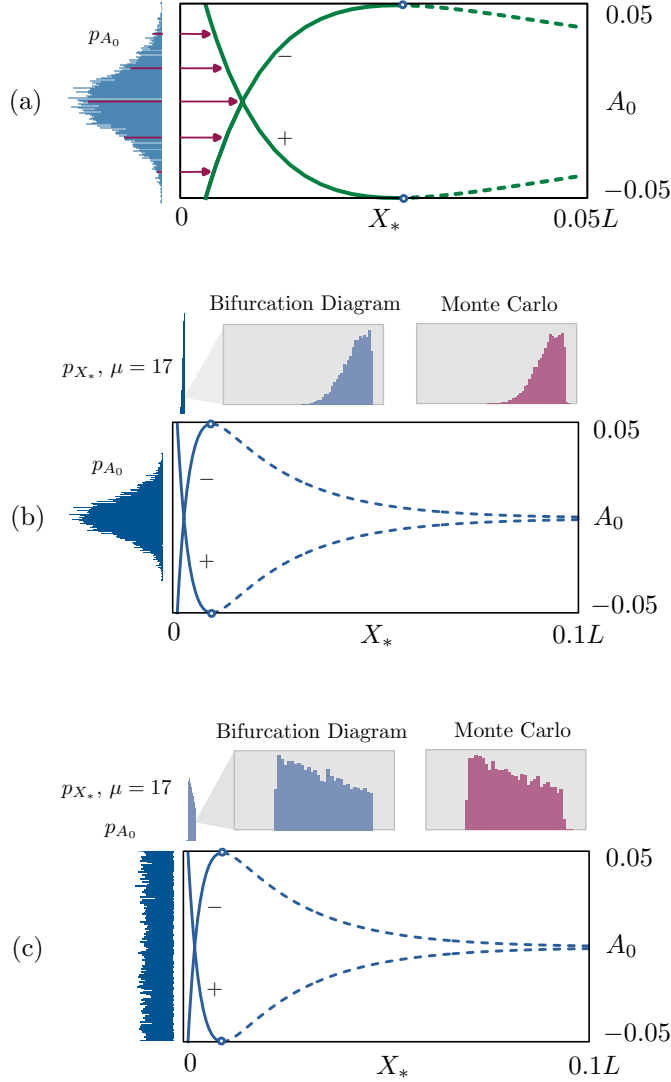


FIG. 19. As for Figure 18 but repeated for $\mu = 17 > \mu_\infty$. (a) At this value of μ the deterministic system possesses two stable solutions that depend on the inlet amplitude A_0 . We label these solutions using the symbols $-$ and $+$ according to $A_*(X_*) > 0$ and $A_*(X_*) < 0$, respectively. These solutions remain stable (solid lines) up to a maximum value of A_0 and are unstable beyond (dashed lines). Since $S(A(x, 0, \omega)) \approx 0$, we expect that system (5.2) will evolve towards X_*^+ when $A_0(\omega) > 0$, and towards X_*^- when $A_0(\omega) < 0$, as indicated by the arrows. (b), (c) The hypothesis of panel (a) is used to approximate p_{X_*} (upper panels); for $\mu = 17$ the front is located very close to the inlet, and the histograms compare favorably with Monte Carlo simulations of (5.2) with $T = 10^3$ and 10^4 samples.

where two stable solutions with different X_* coexist. Both branches are initially stable (solid lines) and destabilise at saddle-node bifurcations; importantly, there are two stable attracting steady states for each $A_0 \in [-0.05, 0.05]$, selected by the initial conditions for the problem (5.2). Since $S(A(x, 0, \omega)) \approx 0$, it is reasonable to expect that the system will be attracted to a solution on the branch with $A_*(X_*) > 0$ when $A_0(\omega) < 0$ (labeled $-$), and to a solution on the branch with $A_*(X_*) < 0$ when

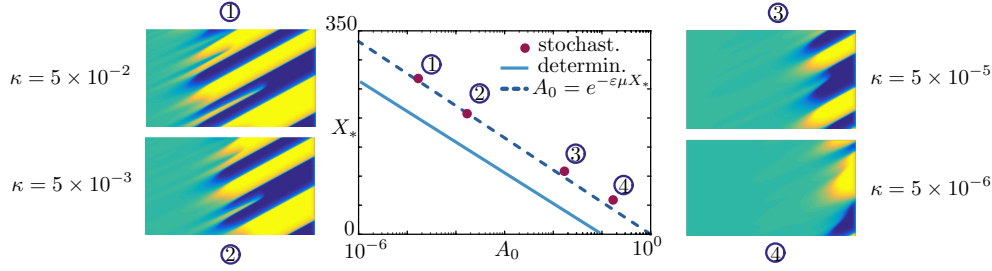


FIG. 20. Direct numerical simulations of system A with a stochastic inlet boundary condition $A_0(t) = \kappa\eta(t)$, where $\kappa > 0$ and $\eta(t)$ is a Wiener process (see Equation (5.3)). We show contour plots of $A(x, t)$ for $(x, t) \in [0, 377] \times [0, 500]$ for $\mu = 6$, $\varepsilon = 1/144$ and various values of κ (panels 1–4). For $\kappa = 0$, the only attracting solution is the trivial steady state $A_* \equiv 0$ but, as the noise is turned on, noise-sustained structures appear. As κ increases so does the inlet amplitude A_0 , resulting in decreased distance X_* to the noise-sustained front. The deterministic and stochastic setting have a common scaling $A_0 = e^{-\varepsilon\mu X_*}$ (see text).

$A_0(\omega) > 0$ (labeled +), as sketched in Figure 19(a). In passing, we note that the X_* scale in Figure 19(a) indicates that the front location is much closer to the inlet, $X_* \in [0, 0.05L]$, for $\mu > \mu_\infty$ than when $\mu < \mu_\infty$ (cf. Figure 18).

The above conjecture is supported by the numerical experiments reported in Figures 19(b,c). We repeated the procedure of Figure 18 for $\mu = 17$ assuming that

$$X_*(A_0) = \begin{cases} X_*^+(A_0) & \text{if } A_0 < 0, \\ X_*^-(A_0) & \text{if } A_0 \geq 0 \end{cases}$$

and found that the resulting densities p_{X_*} are sharply peaked around the inlet (upper panels in (b,c)). The insets show an excellent agreement with the histograms computed via Monte Carlo simulations of the system (5.2) using $T = 10^3$ and 10^4 samples.

5.5. Stochastic simulations. We report, finally, on the results of stochastic simulations of system A in which the inlet value is a continuous stochastic process. Specifically, we set $A(0, t) = \kappa\eta(t)$, where $\kappa > 0$ and $\eta(t)$ is a Wiener process, i.e.,

$$(5.3) \quad \eta(0) = 0 \text{ a.s.,} \quad \eta(t+s) - \eta(t) \stackrel{\text{i.i.d.}}{\sim} \sqrt{s}\mathcal{N}_{0,1}(\cdot), \quad \text{for all } t, s > 0.$$

In the following we use the standard Euler-Maruyama scheme to integrate the resulting stochastic PDE. The results in Figure 20 demonstrate the presence of noise-sustained structures in the regime $0 < \mu < \mu_\infty$, thereby generalizing the results for fixed but randomly selected values of η in Section 5.4 and those for periodically oscillating inlet boundary conditions in Section 5.3. For $\kappa = 0$, the only attracting solution is the trivial steady state $A_* \equiv 0$. As the noise is turned on, noise-sustained structures appear, echoing the behavior found for random but time-independent inlet conditions (Figure 17(b)). In order to make a quantitative comparison with the deterministic case, we compute X_* using the definition (5.1) and we plot this value against $\max_t |A_0(t)|$ for each of the realizations reported in Figure 20.

The $X_*(A_0)$ scaling observed in Figure 20 can be understood completely in terms of the deterministic case with $\kappa = 0$ (solid blue line in Figure 20): this curve, obtained by numerical continuation, is the semi-logarithmic plot of the $X_*(A_0)$ curve in Figure 17(b) for $\mu = 6$. To each deterministic stationary profile $A(x)$ there corresponds

an orbit of the spatial dynamical system A with initial condition close to the origin $(0, 0)$. As noted in Section 4, the origin is unstable with eigenvalues $\lambda_{1,2} \approx \varepsilon\mu, 1$. The profiles $A(x)$ of interest (see, for instance, panels 6, 7, 8 of Figure 15) correspond to orbits that spend a long “time” close to the origin, following the repelling slow manifold: these orbits drift along the weak eigendirection of the origin, before being ejected and crossing the threshold $|A(x)| = 0.1$. Since

$$A(x) \approx A_0 e^{\varepsilon\mu x}, \quad x \in [0, 0.1],$$

it follows that X_* is determined by the condition $0.1 \approx A_0 e^{\varepsilon\mu X_*}$ which, as seen in Figure 20, represents the common scaling between the stochastic and deterministic cases.

6. Discussion. The problem as formulated here contains two independent parameters, the forcing parameter μ responsible for the primary instability and the advection velocity U . Similar situations arise in Rayleigh-Bénard convection with throughflow. This problem has been studied by a number of authors, both for transverse rolls [23] and longitudinal rolls [24]. However, in both these cases the pattern-forming instability is supercritical. This is also the case for Taylor vortices in a Taylor-Couette system with axial flow [26]. Related work on binary fluid convection [2] studies a subcritical system but with periodic boundary conditions, thereby eliminating the possible influence of inlet perturbations. The effect of such perturbations is usually studied within a linearized theory [9] or via numerical simulations, either of model equations [9, 10, 6] or the full partial differential equations [20, 24]. However, in no case has the full bifurcation diagram been computed and the interaction between type I and type II solutions investigated.

In contrast, in shear flow instability, including plane Poiseuille (channel) flow and circular Poiseuille (pipe) flow, the shear flow is simultaneously responsible for the presence of finite amplitude instability of the laminar state and the advection of the resulting turbulent structures downstream [11, 15]. Such flows are therefore fundamentally different from the problem studied here, and indeed in carefully controlled pipe flow experiments the laminar state persists to large values of the Reynolds number ($Re \sim 60\,000$ [15]), the parameter that simultaneously quantifies both the shear rate and the advection velocity, despite the inevitable presence of inlet fluctuations. For comparison, finite amplitude perturbations trigger pipe flow turbulence at $Ra \sim 2\,000$.

We mention that Eq. (1.1), in the form $A_t = A_{xx} - (U - \alpha|A|^2)A_x + \mu A - |A|^2 A$, may also be considered to be an amplitude equation for the complex amplitude (x, t) of a wavetrain of the form $A(x, t) \exp ikx$. The real subspace of this equation is invariant and identical to Eq. (1.1). The stationary solutions to this BVP constructed here therefore remain solutions of this higher-dimensional system; the zeros of $A(x)$ now correspond to phase defects – locations where the phase of the solution jumps by π , and it is possible that the breakup of a wavetrain into domains with different phase speeds observed in simulations of the complex Ginzburg-Landau equation with drift [29, 16] can be understood along these lines.

In summary, we expect that the ideas presented in this work may find application in different areas of fluid mechanics where a strongly subcritical instability interacts with an imposed flow.

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7. Appendix: Numerical methods. This appendix summarizes the numerical technique used to obtain the results reported in this paper.

We discretize the PDE (2.1) using a grid of evenly spaced points $x = \{x_i\}_{i=1}^{n+1}$, with $x_i = iL/n$, and constructing the approximation vectors $A(t) = \{A_i(t)\}$, where $A_i(t) \approx A(x_i, t)$. We employ the method of lines and obtain a large set of nonlinear ODEs

$$\begin{aligned} A_1 &= A_0, \\ (7.1) \quad \dot{A}_i &= \varepsilon \mu A_i - \sqrt{\varepsilon} A_i^3 + \sum_{j=1}^{n+1} \left[(D_x^2)_{ij} - (1 - \alpha A_i^2)(D_x)_{ij} \right] A_j \quad i = 2, \dots, n, \\ A_{n+1} &= 0, \end{aligned}$$

where D_x and D_x^2 are $(n-1) \times (n+1)$ finite-difference differentiation matrices for the operators ∂_x and ∂_{xx} , respectively. In all PDE computations we use $n = 4000$ gridpoints. Time stepping is performed using MATLAB's `ode23s` with default parameters and explicitly formed Jacobian. For the stochastic time stepping we use a standard Euler-Maruyama scheme with $dt = 10^{-3}$. Steady states are computed using MATLAB's `fsolve` with default parameter values and continued in parameter space using the routines developed in [25]. Stability of a steady state $A_* = \{A_*(x_i)\}$ is studied by computing the first 40 eigenvalues of the problem $M\psi = J(A_*)\psi$ with the smallest absolute value, where

$$M = \text{diag}(0, 1, \dots, 1, 0) \in \mathbb{R}^{(n+1) \times (n+1)}$$

$$J = \begin{bmatrix} D_x^2 - \text{diag}(1 - \alpha A_*^2) D_x + \text{diag}(2\alpha A_* D_x A_* + \varepsilon \mu - 3\sqrt{\varepsilon} A_*^2) \\ e_{n+1} \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}$$

and the $\{e_i\}$ are row vectors forming the canonical basis of \mathbb{R}^{n+1} .

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